

RIBBON DISKS WITH THE SAME EXTERIOR

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ABSTRACT. We construct an infinite family of slice disks with the same exterior, which gives an affirmative answer to an old question asked by Hitt and Sumners in 1981. Furthermore, we prove that these slice disks are ribbon disks.

1. INTRODUCTION

One of the most outstanding problems in knot theory has been whether knots are determined by their complements or not. The celebrated theorem of Gordon and Luecke [11] states that if the complements of two classical knots in the 3-sphere S^3 are diffeomorphic, then these knots are equivalent (for more recent results, see [7, 18, 24]). For higher-dimensional knots, there exist at most two inequivalent n -knots ($n \geq 2$) with diffeomorphic exteriors, see [4, 8, 19]. Examples of such n -knots are given in [5] and [10].

We consider an analogous problem for slice disks, that is, smoothly and properly embedded disks in the standard 4-ball B^4 . The situation is quite different. Let X be the exterior of a slice disk, and define $\zeta(X)$ to be the number of inequivalent slice disks whose exteriors are diffeomorphic to X . In 1981, Hitt and Sumners [14, Section 4] asked the following.

Question 1. Is there a slice disk exterior X with $\zeta(X) = +\infty$?

It seems that no essential progress has been made to Question 1 until recently. One of the reasons is that when we consider Question 1, we often encounter the smooth Poincaré conjecture in dimension four, which is one of the most challenging unsolved problems. In 2015, Larson and Meier [20] produced infinite families of distinct homotopy-ribbon disks with homotopy equivalent exteriors, which gives a partial answer to Question 1. In this paper, we prove the following.

Theorem 1.1. *There exist infinitely many distinct slice disks with the same exterior, which provides an affirmative answer to Question 1. Furthermore, these slice disks are chosen to be ribbon disks.*

Here we give a historical remark. In [14, Section 4], Hitt and Sumners also asked whether there exist infinitely many higher-dimensional slice disks with the same exterior or not. Note that this question already had been solved. Actually, after pioneering work [14, 15, 23], Suciuc [25] proved that there exist infinitely many distinct n -ribbon disks with the same exterior for $n \geq 3$ in 1985, and Question 1 was remained open.

This paper is organized as follows: In Section 2, we recall some basic definitions. In Section 3, we recall a construction of slice disks, and prove the first half of Theorem 1.1. In Section 4, we prove the latter half of Theorem 1.1. In Section 5, we give some remarks.

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2. BASIC DEFINITIONS

In this short section, we recall some basic definitions and background.

We define a *slice disk* to be a smoothly and properly embedded disk $D \subset B^4$, and the boundary of D , $\partial D \subset S^3$, is called a *slice knot*. A knot $R \subset S^3$ is called *ribbon* if it bounds an immersed disk $\Delta \subset S^3$ with only ribbon singularities. For the definition of a ribbon singularity, see the left picture of Figure 1. By pushing the interior of Δ into B^4 , we obtain a slice disk whose boundary is R , and the resulting slice disk is called a *ribbon disk*. It is well known that this ribbon disk is uniquely determined by $\Delta \subset S^3$ up to isotopy. The *slice-ribbon conjecture*, also known as the ribbon-slice problem, states that every slice knot is a ribbon knot, namely, it always bounds a ribbon disk. Our work is partially motivated by this conjecture¹ Here we give an example of a ribbon knot.

Example 2.1. *The knot in the middle picture of Figure 1 is a ribbon knot since it bounds an immersed disk $\Delta \subset S^3$ as in the right picture of Figure 1.*

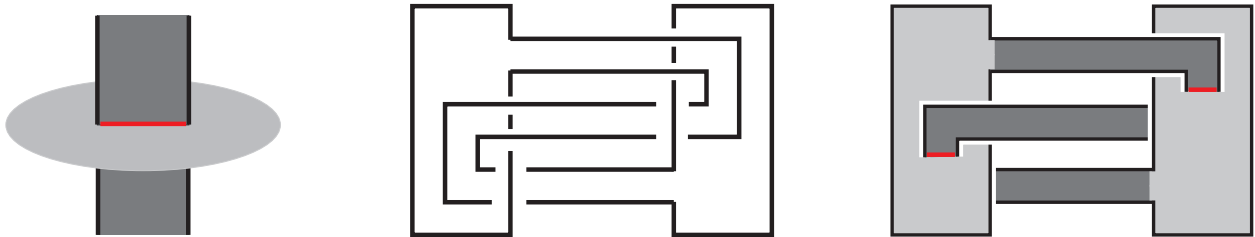


FIGURE 1. A ribbon singularity (colored red), a ribbon knot, and an immersed disk $\Delta \subset S^3$.

Two knots K_1, K_2 are *equivalent* if there exists a diffeomorphism $f: S^3 \rightarrow S^3$ such that $f(K_1) = K_2$. The *exterior* of a knot K is the 3-manifold $S^3 \setminus \nu(K)$, where $\nu(K)$ is an open tubular neighborhood of K in S^3 . It is unique up to diffeomorphisms, and its interior is diffeomorphic to the complement of K . Similarly two slice disks D_1, D_2 are *equivalent* if there exists a diffeomorphism $g: B^4 \rightarrow B^4$ such that $g(D_1) = D_2$, and the *exterior* of a slice disk D is the 4-manifold $B^4 \setminus \nu(D)$, where $\nu(D)$ is an open tubular neighborhood of D in B^4 . Note that (ambient) isotopic two slice disks D_1 and D_2 are equivalent.

3. A CONSTRUCTION OF SLICE DISKS

In this section, we give a construction of slice disks in B^4 . This construction is a variant of Hudson and Sumners's one (see [6], [16]).

¹Hass [12] proved that a slice disk is a ribbon disk if and only if it is isotopic to a minimal disk in B^4 , see also [13, Appendix B], [27, p450]. Therefore the slice-ribbon conjecture might be solved (affirmatively or negatively) using geometric analysis or geometric measure theory.

Let $U \sqcup L_1 \sqcup \cdots \sqcup L_m$ be the $(m+1)$ -component unlink. By definition, it bounds disks $\Delta \sqcup \Delta_1 \sqcup \cdots \sqcup \Delta_m$ in S^3 . Push the interiors of the spanning disks into B^4 , then we obtain smoothly and properly embedded disks in B^4 . We denote it by

$$D \sqcup \tilde{\Delta}_1 \sqcup \cdots \sqcup \tilde{\Delta}_m,$$

where $\partial D = U$ and $\partial \tilde{\Delta}_i = L_i (i = 1, \dots, m)$. Let X be the 4-manifold obtained from B^4 by removing tubular a neighborhood of $\tilde{\Delta}_1 \sqcup \cdots \sqcup \tilde{\Delta}_m$. Note that X depends only on the isotopy type of the m -component unlink $L_1 \sqcup \cdots \sqcup L_m$, and usually it is denoted by the link with dots.

Next, attach 2-handles $h_i^2 (i = 1, \dots, m)$ along circles in

$$S^3 \setminus (U \sqcup L_1 \sqcup \cdots \sqcup L_m) \subset \partial X$$

so that the handle diagram of the resulting 2-handlebody $X \cup h_1^2 \cup \cdots \cup h_m^2$ is isotopic to that in Figure 2, where γ_i is the framing of the 2-handle $h_i^2 (i = 1, \dots, m)$. (In this paper, a *2-handlebody* means a 4-manifold which consists of a 0-handle, 1-handles, and 2-handles.)

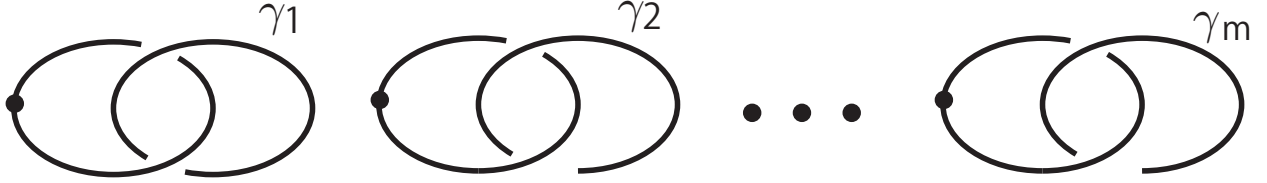


FIGURE 2. A handle diagram of $X \cup h_1^2 \cup \cdots \cup h_m^2$.

By the handle canceling theorem, the 2-handlebody $X \cup h_1^2 \cup \cdots \cup h_m^2$ is diffeomorphic to B^4 . Thus we obtain a new slice disk

$$D \subset X \cup h_1^2 \cup \cdots \cup h_m^2 \approx B^4,$$

where the symbol \approx stands for a diffeomorphism. Here we give a schematic picture of this construction for the case $m = 1$ in Figure 3.

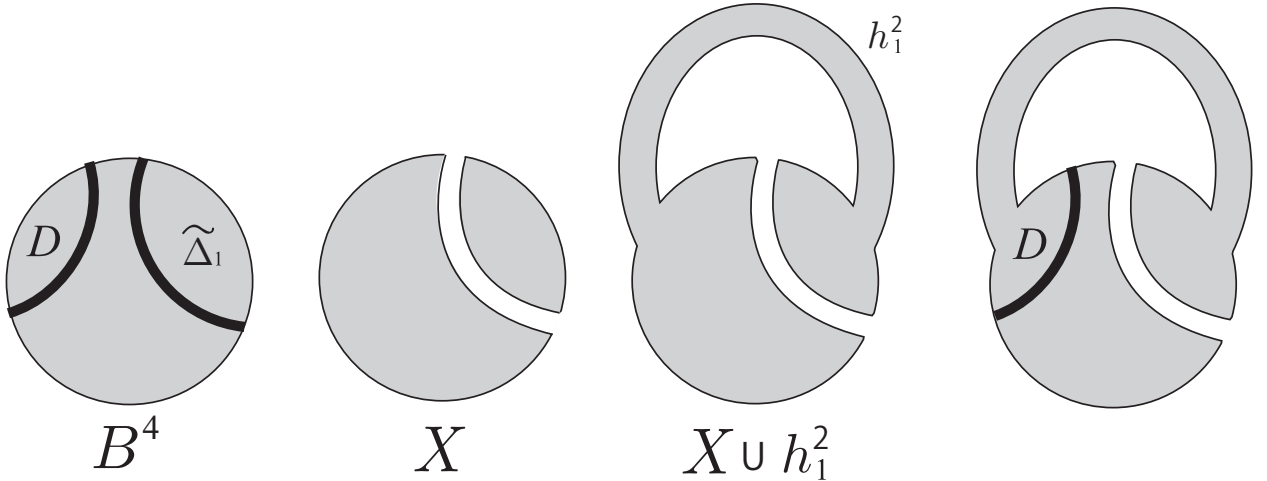


FIGURE 3. $D \sqcup \tilde{\Delta}_1$ in B^4 , the 4-manifold X , the 2-handlebody $X \cup h_1^2$ which is diffeomorphic to B^4 , and the new slice disk D in $X \cup h_1^2 \approx B^4$.

We give a concrete example of this construction for the case $m = 2$.

Example 3.1. Let $U \sqcup L_1 \sqcup L_2$ be the 3-component unlink, and $\Delta \sqcup \Delta_1 \sqcup \Delta_2$ its spanning disks in S^3 as in Figure 4. Push the interiors of the spanning disks into B^4 , then we obtain smoothly and properly embedded disks in B^4 . We denote it by

$$D \sqcup \tilde{\Delta}_1 \sqcup \tilde{\Delta}_2,$$

where $\partial D = U$, $\partial \tilde{\Delta}_1 = L_1$ and $\partial \tilde{\Delta}_2 = L_2$. Let X be the 4-manifold obtained from B^4 by removing a tubular neighborhood of $\tilde{\Delta}_1 \sqcup \tilde{\Delta}_2$, see the right picture of Figure 4.



FIGURE 4. The 3-component unlink $U \sqcup L_1 \sqcup L_2$, its spanning disks $\Delta \sqcup \Delta_1 \sqcup \Delta_2$ in S^3 , and the handle diagram of X .

Let n be an integer. Attach two 2-handles h_1^2, h_2^2 as in the left picture of Figure 5, where the framing of h_1^2 ($= \gamma_1$) is $-n$ and that of h_2^2 ($= \gamma_2$) is n . Notice that the attaching circles of the 2-handles h_1^2, h_2^2 intersect the interior of $\Delta \subset S^3$.

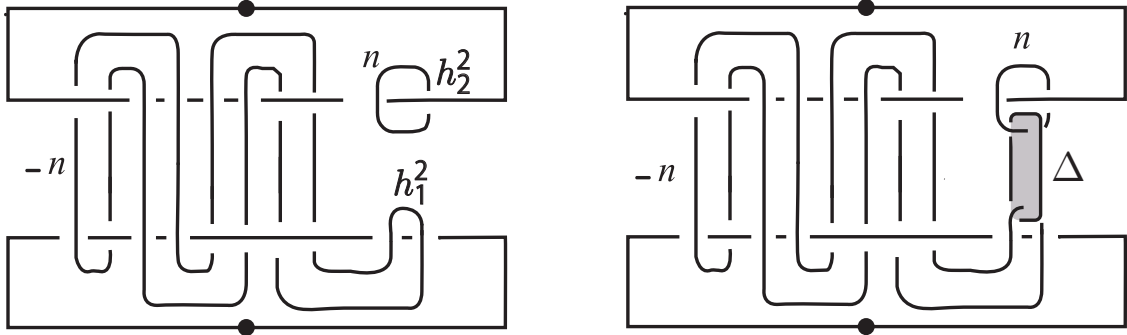


FIGURE 5. Two 2-handles h_1^2, h_2^2 , and the disk Δ in the handle diagram of $X \cup h_1^2 \cup h_2^2$.

By the construction, the 2-handlebody $X \cup h_1^2 \cup h_2^2$ is diffeomorphic to B^4 . Then we obtain a slice disk

$$D \subset X \cup h_1^2 \cup h_2^2 \approx B^4,$$

which is also obtained by pushing the interior of the shaded disk Δ in the right picture of Figure 5. This slice disk is indexed by n since attaching maps of $h_1^2 \cup h_2^2$ depend on n , and we denote it by D_n . This slice disk D_n is used in the proof of Theorem 1.1.

We are ready to prove the first half of Theorem 1.1.

Proof of the first half of Theorem 1.1. Let D_n ($n \in \mathbb{Z}$) be the slice disk in Example 3.1, and X_n the exterior of D_n . By the definition of dotted circles, X_n is represented by the picture in Figure 6. Then X_n is diffeomorphic to X_0 , which implies that the exteriors of D_n are the same up to diffeomorphisms. This result follows from the well-known fact that two handle diagrams in Figure 7 are related by a sequence of handle moves, see [3, 9]. Now all we have

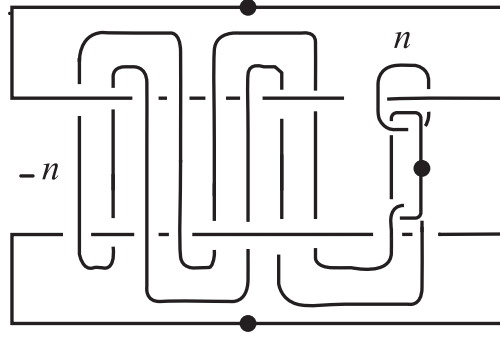
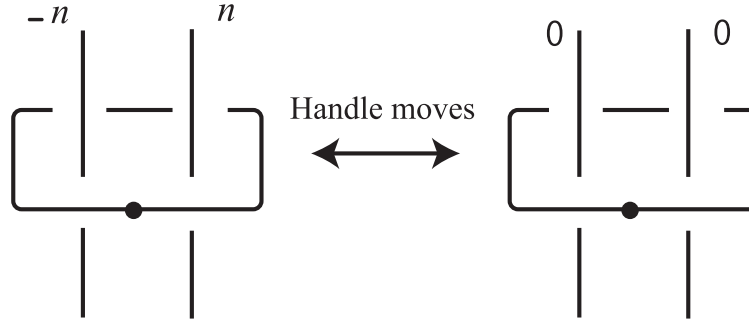
FIGURE 6. A handle diagram of X_n .

FIGURE 7. Two handle diagrams related by a sequence of handle moves.

to do is to distinguish slice disks D_n . Recently, Takioka [26] calculated the Γ -polynomial² of the knots ∂D_n . More precisely, he showed that the span of the Γ -polynomial of ∂D_n is $2n + 2$ and proved that the knots ∂D_n are mutually distinct for $n \geq 0$. Therefore slice disks D_n are mutually distinct for $n \geq 0$. \square

Remark 3.2. The knot ∂D_0 is $4_1 \# 4_1$, that is, the connected sum of two figure-eight knots. We can prove that ∂D_n is isotopic to ∂D_{-n} (by using the symmetry of $4_1 \# 4_1$).

Remark 3.3. Here we give an alternative proof that the exteriors of D_n , denoted by X_n , are the same up to diffeomorphisms. By a handle slide in the left picture of Figure 8, we obtain the right picture of Figure 8 after canceling the $1/2$ -canceling pair. Therefore 4-manifolds

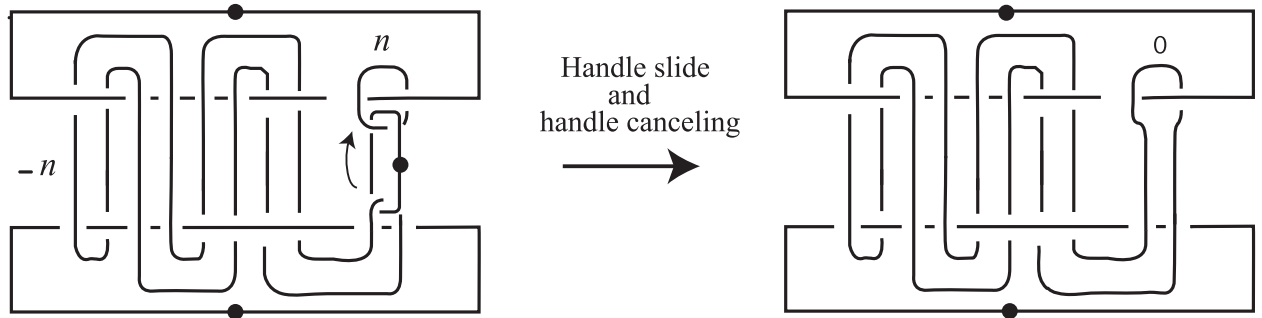


FIGURE 8.

² This is a polynomial invariant introduced by Akio Kawauchi in [17], which is a specialization of the Homflypt polynomial. This polynomial is independent of the Alexander polynomial and the Jones polynomial, and experientially easy to calculate (though we do not know its computational complexity).

X_n are the same up to diffeomorphisms.

Note that X_n is diffeomorphic to the exterior of the ribbon disk represented by the picture in Figure 9 (see [3, Subsection 1.4] or [9, Subsection 6.2]). This fact does not immediately imply that D_n is a ribbon disk. However, in this case, D_n is a ribbon disk as proven later.

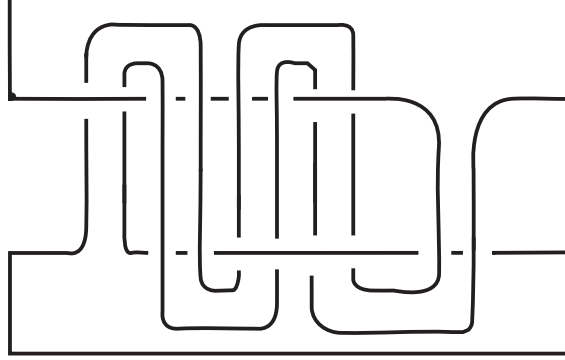


FIGURE 9. A ribbon knot, which determines the corresponding ribbon disk.

We conclude this section by asking the following question.

Question 2. Let D be a slice disk whose exterior is diffeomorphic to a ribbon disk exterior. Then is D a ribbon disk ?

4. PROOF OF THE LATTER HALF OF THEOREM 1.1

In this section, we prove the latter half of Theorem 1.1.

The following two lemmas are important when we deal with ribbon disks in terms of handle diagrams of B^4 .

Lemma 4.1. *Let F and F' be two smooth disks in handle diagrams of B^4 as shown in Figure 10. Then the corresponding slice disks in B^4 obtained from F and F' are isotopic.*

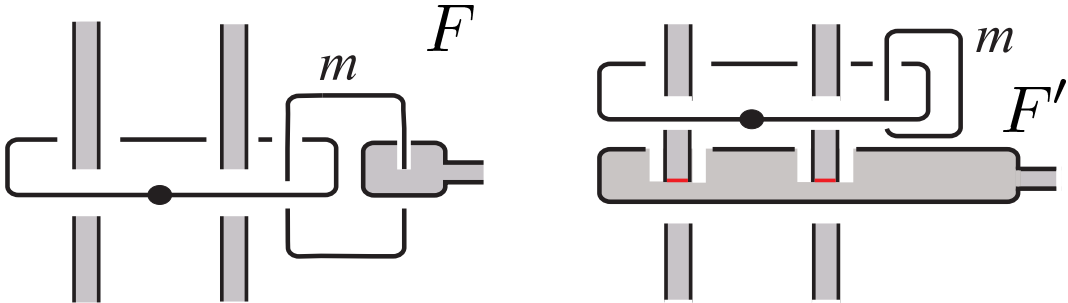
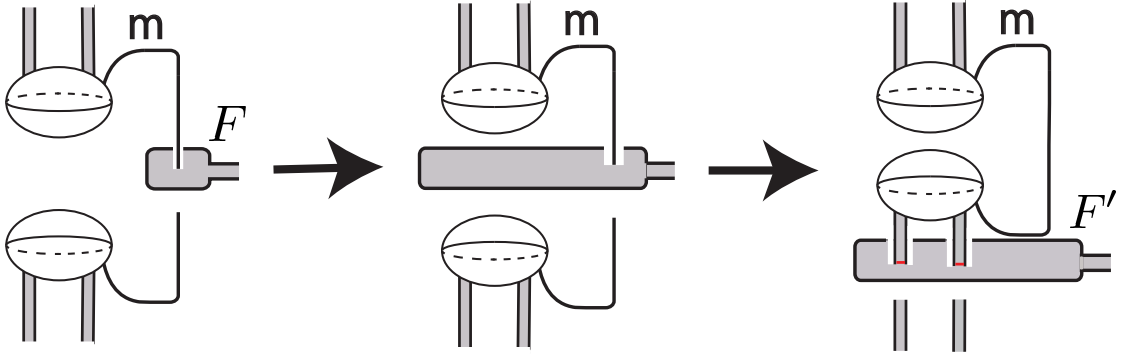
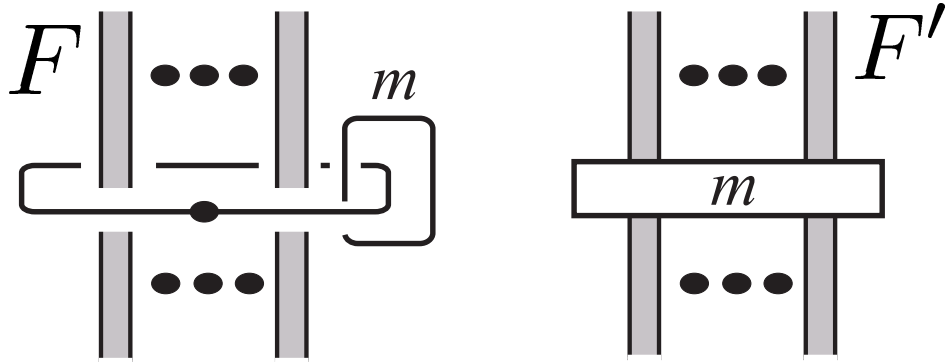


FIGURE 10. Two smooth disks F and F' in handle diagrams of B^4 .

Proof. See Figure 11. □

Lemma 4.2. *Let F and F' be two smooth disks in handle diagrams of B^4 as shown in Figure 12. Then the corresponding slice disks in B^4 obtained from F and F' are isotopic.*

Proof. After deforming F to the 2-handle in the left picture of Figure 12, we cancel the $1/2$ -handle pair. Then we obtain the right picture of Figure 12. □

FIGURE 11. A relation between F and F' .FIGURE 12. Two disks F and F' . Here the box named m means the m -full twists.

We are ready to prove the latter half of Theorem 1.1.

Proof of the latter half of Theorem 1.1. Let D_n ($n \in \mathbb{Z}$) be the slice disk in Example 3.1. We will prove that D_n is a ribbon disk. Recall that D_n is obtained by pushing the interior of the disk Δ in the left picture of Figure 13. We will consider Δ instead of D_n .

First, we deform the disk Δ as in Figure 13, and denote the resulting disk by Δ again. Next, we deform the disk Δ as in Figure 14, and denote the resulting disk by Δ' . Note that

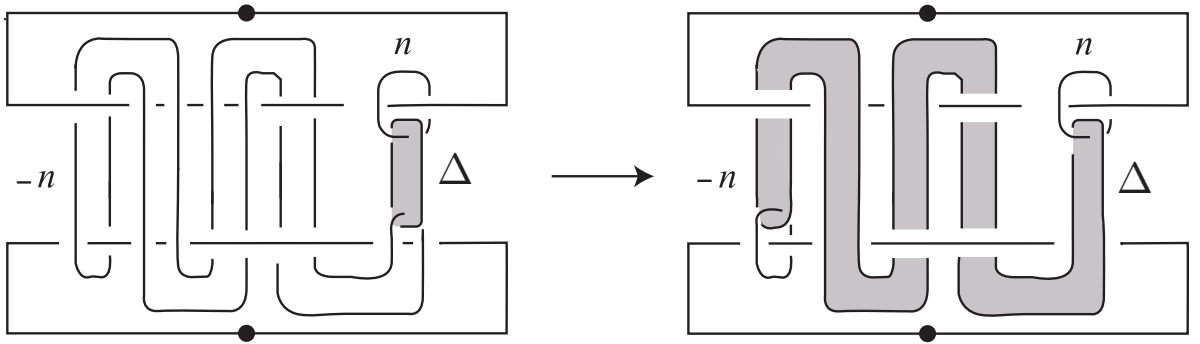
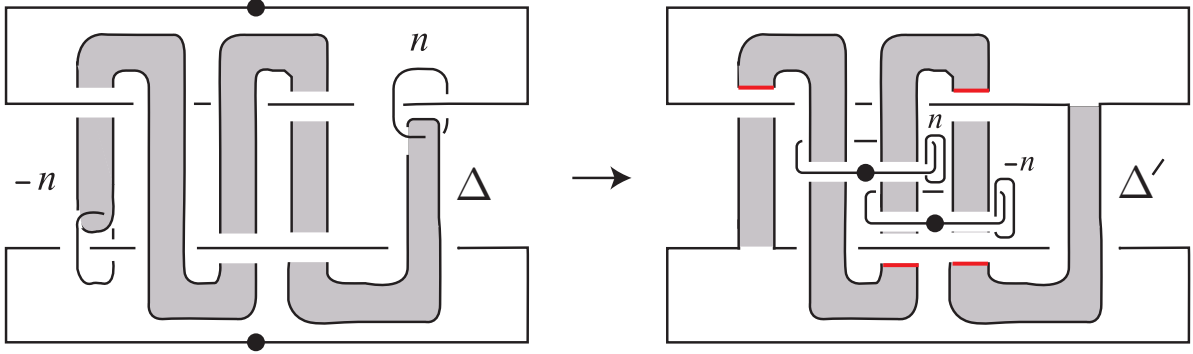


FIGURE 13.

the disk Δ' has four ribbon singularities, however, we do not draw the whole picture of Δ' for simplicity. By Lemma 4.1, the corresponding slice disks are isotopic.

FIGURE 14. Δ' is an immersed disk with four ribbon singularities.

Finally, we deform the disk Δ' as in Figure 15, and denote the resulting disk by Δ'' . By Lemma 4.2, the corresponding slice disks are isotopic. As a summary, two slice disks obtained from Δ and Δ'' are isotopic. The slice disk obtained from Δ'' is a ribbon disk, see

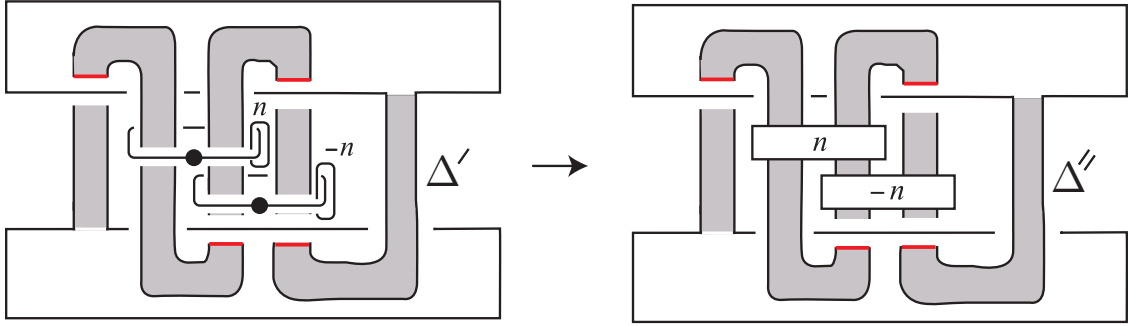


FIGURE 15.

the right picture of Figure 15. Therefore D_n (which obtained from Δ) is a ribbon disk. \square

Remark 4.3. The right picture of Figure 15 tells us that the knot ∂D_n is isotopic to that in Figure 16. This implies that ∂D_n is obtained from the ribbon knot $\partial D_0 = 4_1 \# 4_1$ by

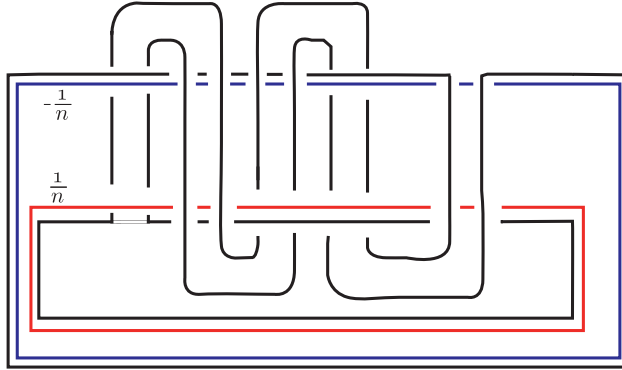


FIGURE 16.

Osoinach's construction (annular twisting construction) introduced in [22], see also [1, 2]. Furthermore, it turns out that the knots ∂D_n are the same as those in [22, Figure 7], however, we omit the proof since this fact is not used in this paper.

5. SOME REMARKS

We overlooked in [2] that an affirmative answer to Question 1 follows from the main result in [2]. We outline this argument below.

For the ribbon knot 8_{20} , we can construct infinite family of knots K_n via the annular twisting construction, and then K_n bounds a disk E_n in a homotopy 4-ball W_n , see [2]. We proved that W_n is diffeomorphic to B^4 , implying that E_n is a slice disk [2, Theorem 3.1]. By the construction, it is straightforward to see that the exteriors of E_n are the same (up to diffeomorphisms), implying an affirmative answer to Question 1. Note that the argument in this paper is much easier than that in [2].

Recently, Miller and Piccirillo [21] pointed out that the fact that K_n is a slice knot, that is, it bounds a certain disk E'_n in B^4 directly follows from [1]. We do not know whether slice disks E_n and E'_n are isotopic or not. Therefore we do not know whether the exteriors of E'_n are the same or not (up to diffeomorphisms). In [2, Theorem 5.4], we also proved that K_n is a ribbon knot, where we used the following lemma ([2, Lemma 5.1]).

Lemma 5.1. *Let HD be a handle diagram of B^4 . Suppose that HD is changed into the empty handle diagram of B^4 by handle slides, adding or canceling $1/2$ -handle pairs. Then the belt sphere of any 2-handle of HD is a ribbon knot.*

Note that we did NOT prove that E_n is a ribbon disk. To prove this, it is enough to show the following.

Claim. Let HD be a handle diagram of B^4 . Suppose that HD is changed into the empty handle diagram of B^4 by handle slides, adding or canceling $1/2$ -handle pairs. Then the co-core disk of any 2-handle of HD is a ribbon disk.

This claim will be proved in a forthcoming paper.

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